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Finitary objects and ultrapowers

by

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Introduction. When we deal with categories of systems with structures, it often looks desirable to set up a notion that distinguishes algebraic structures, such as ordered sets, groups, modules etc., from infinitistic theories, such as topological spaces, complete lattices etc. One attempt was made in [4] and studied particularly in connection with ultrapowers defined in terms of categories. However, only concrete categories were treated there. The purpose of this paper is to generalize the notion of finitary objects defined in [4] to abstract categories.

In §1, we define the basic notions about finitary objects and study some of their properties. Particularly, the following two theorems are proved. First, in a category with proper completeness, the full subcategory generated by all finitary objects is closed with left limits of small diagrams (Theorem 1) and extremal subobjects (Theorem 2). Second, being finitary is, in a way, an intrinsic property of objects, and, if an object is finitary in a left complete category, then it is also finitary in any extension of it, in which it is left closed (Theorem 3).

In §2, one of the main theorems in [4], that states that, in a complete category, if the diagonal map from an object A to its ultrapower is always an embedding, then A must be finitary, is generalized (Theorem 4). It is not only the version of the theorem in the new definition, but the result is somewhat improved.

As for the terminology, we mainly followed Isbell [2], and the completeness of a category is understood in his sense.

§1. Finitary Objects. Let $\underline{C}$ be a general abstract category and $Ob(\underline{C})$ the collection of all objects in $\underline{C}$.

Definition 1. Let $\{b_\lambda\colon X_\lambda \to B \mid \lambda \in \Lambda\}$ be a set of coterminal morphisms in $\underline{C}$. This set is said to *cover* B, if there are no extremal monomorphisms —i.e., monomorphisms with no proper initial epifactors, cf. [2]—that factors all b_λ, $\lambda \in \Lambda$, at the same time, save for isomorphisms. The set $\{b_\lambda\colon X_\lambda \to B \mid \lambda \in \Lambda\}$ is called *compatible with* another $\{a_\lambda\colon X_\lambda \to A \mid \lambda \in \Lambda\}$, if there exists an $f\colon B \to A$ such that $a_\lambda = f b_\lambda$ for all $\lambda \in \Lambda$. The former is called *finitely compatible with* the latter, if, for any finite subset M of Λ, there exists an $f_M\colon B \to A$ such that $a_\lambda = f_M b_\lambda$ for every $\lambda \in M$. A is called *finitary under* B, if for any sets of morphisms $\{b_\lambda: X_\lambda \to B \mid \lambda \in \Lambda\}$ and $\{a_\lambda: X_\lambda \to A \mid \lambda \in \Lambda\}$, the former is compatible with the latter, provided the former covers B and is finitely compatible with the latter. A is called *finitary* if it is finitary under every object in $\underline{C}$.

Example. In the category of topological spaces, even a finite space is not finitary save for the singleton set. Let N be the discrete set of all natural numbers, $N^\dagger$ the set N with an additional element p to which the increasing sequence of $n \in N$ converges and $X = \{x, y\}$ a two point set. For $n \in N$, let $b_n: X \to N^\dagger$ be such that $b_n(x) = n$ and $b_n(y) = p$ while $a_n: X \to X$ the identity for every $n \in N$. Then $\{b_n \mid n \in N\}$ covers $N^\dagger$ and finitely compatible with $\{a_n \mid n \in N\}$, but not compatible.

The same example with modifications of interpretation shows that in the category of complete lattices with morphisms of supremum-preserving mappings, a two-point set $X = \{x, y\}$ with $x < y$ is not finitary.

Remark 1. It is easily checked, as is the intention of the definition, that in a category of relational systems in which all defining relations are finitary, every object is finitary. Thus, in the categories of ordered sets, groups, algebraic systems etc., every object is finitary.

However, observe the following example: Let N be the ordered set of all natural numbers with the natural ordering, $N^\dagger$ the set N with an additional greatest element p and $X = \{x\}$ a singleton set. For $n \in N$, let $b_n: X \to N^\dagger$ and $a_n: X \to N$ be such that $b_n(x) = a_n(x) = n$. Then in the category of ordered sets with morphisms of order-preserving mappings, the set $\{b_n \mid n \in N\}$ is finitely compatible with $\{a_n \mid n \in N\}$ but not compatible. This is the case when the former does not cover $N^\dagger$.

Remark 2. In a category which has equalizers, for a set $\{b_\lambda: X_\lambda \to B \mid \lambda \in \Lambda\}$ that covers B and is compatible with $\{a_\lambda: X_\lambda \to A \mid \lambda \in \Lambda\}$, the morphism $f: B \to A$ such that $a_\lambda = f b_\lambda$ for every $\lambda \in \Lambda$ is uniquely determined. Indeed, if $f': B \to A$ is another such, then the equalizer $m: B' \to B$ of f and f' is an extremal monomorphism that factors all b_λ.

As for extremal monomorphisms we recall the following:

LEMMA 1. *In a category* $\underline{C}$ *which has pushouts;*

(i) *if the square below is a pullback diagram with an extremal monomorphism* $Z \to W$, *then* $X \to Y$ *is also an extremal monomorphism,*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

(ii) *the intersection of a family of extremal subobjects of an object is an extremal subobject of the object, and*

(iii) *the composition of two extremal monomorphisms is an extremal monomorphism.*

The proofs are seen in [4].

THEOREM 1. *In a category $\underline{C}$ which has equalizers, the full subcategory $\underline{F}$ of $\underline{C}$ generated by all of its finitary objects is left small-closed in $\underline{C}$; that is, if $\{f_{\alpha\beta\gamma}: A_\alpha \to A_\beta\}_{\alpha\beta\gamma}$ is a small diagram in $\underline{C}$, in which all components A_α are finitary, then its left limit A, if exists, is also finitary.*

Proof. Assume that $\{b_\lambda: X_\lambda \to B \mid \lambda \in \Lambda\}$ covers B and finitely compatible with $\{a_\lambda: X_\lambda \to A \mid \lambda \in \Lambda\}$. Then, letting $\pi_\alpha: A \to A_\alpha$ be the canonical projection to the components, the former is also finitely compatible with $\{\pi_\alpha a_\lambda: X_\lambda \to A_\alpha \mid \lambda \in \Lambda\}$. Since A_α is finitary, there exists a $g_\alpha: B \to A_\alpha$ such that $\pi_\alpha a_\lambda = g_\alpha b_\lambda$ for all $\lambda \in \Lambda$. By the uniqueness of such g_α (Remark 2), we have $g_\beta = f_{\alpha\beta\gamma} g_\alpha$ for any α, β, γ. Hence a $g: B \to A$ is determined so that $g_\alpha = \pi_\alpha g$ for every α. But $\pi_\alpha a_\lambda = \pi_\alpha g b_\lambda$ for every α implies $a_\lambda = g b_\lambda$ and the existence of such g was to be proved. q.e.d.

COROLLARY. *In any category the direct product of finitary objects is also finitary.*

Proof. In the proof of Theorem 1, the existence of equalizers was used to establish the uniqueness of $g_\alpha: B \to A_\alpha$ which again infers the commutativity $g_\beta = f_{\alpha\beta\gamma} g_\alpha$. By lack of such bond-morphisms $f_{\alpha\beta\gamma}$ in the diagram for a direct product, we can omit the part of the proof together with the assumption that the category has equalizers. q.e.d.

THEOREM 2. *If $\underline{C}$ has pullbacks and pushouts, then an extremal subobject of a finitary object is finitary.*

Proof. Let $m: A \to A'$ be an extremal monomorphism to a finitary object A′. Assume that $\{b_\lambda: X_\lambda \to B \mid \lambda \in \Lambda\}$ covers B and is finitely compatible with $\{a_\lambda: X_\lambda \to A \mid \lambda \in \Lambda\}$. Then the former is also finitely compatible with $\{m a_\lambda: X_\lambda \to A' \mid \lambda \in \Lambda\}$. Since A' is finitary, there exists an $f: B \to A'$ such that $m a_\lambda = f b_\lambda$ for all $\lambda \in \Lambda$. Let the square CAA′B in the diagram

$$
\begin{tikzcd}
X_\lambda \arrow[dr, "c_\lambda"] \arrow[drr, "b_\lambda"] \arrow[ddr, "a_\lambda"'] & & \\
& C \arrow[r, "b"'] \arrow[d, "g"] & B \arrow[d, "f"] \\
& A \arrow[r, "m"'] & A'
\end{tikzcd}
$$

be a pullback diagram, then there exists a morphism c_λ such that $b_\lambda = bc_\lambda$ and $a_\lambda = gc_\lambda$. Thus $b: C \to B$ is an extremal monomorphism (see Lemma 1 (i)) that factors all b_λ, $\lambda \in \Lambda$. Hence b is an isomorphism, and we have $a_\lambda = (gb^{-1})b_\lambda$ for every $\lambda \in \Lambda$. q.e.d.

Here we put a few more preliminary remarks.

Let X and Y be objects in $\underline{C}$. X is said to *co-generate* Y, if there is no proper epimorphism $Y \to \cdot$ that factors all morphisms in $\underline{C}(Y, X)$. This is the dual notion of Grothendieck's generator [1]. A category $\underline{C}$ is called *locally small*, if $\underline{C}(Y, X)$ is small for every $X, Y \in Ob(\underline{C})$ (see [2]). The following lemma is also found in [2] in the dual form.

LEMMA 2. *When $\underline{C}$ is locally small and has direct products, X co-generates Y if and only if Y is an extremal subobject of a direct power of X.*

In the discussion below, we sometimes require the existence of (representable) co-intersections of quotient obiects of an object. Though it is not very strong condition, in case that smaller procedure is preferable, we prepare the notion of image-decompositions of morphisms. For a morphism f, the decomposition $f = gh$ with an epimorphism h and an extremal monomorphism g, if exists, is called the *image-decomposition* of f. When the category has pushouts, the image-decomposition is unique for f up to equivalence. If every morphism in the category has an image-decomposition, we say that the category has image-decompositions.

LEMMA 3. *If either* (i) *the category has co-equalizers and co-intersections of quotient objects of any object, or* (ii) *it has pushouts, equalizers and intersections of extremal subobjects of any object, then it has image-decompositions.*

Proof. For $f: X \to Y$, in case (i) the co-intersection of all epimorphisms $X \to \cdot$ that factor f serves as the epifactor of the image-decomposition of f, and in case (ii) the intersection of all extremal monomorphisms $\cdot \to Y$ that factor f serves as the extremal monofactor of the image-decomposition of f. q.e.d.

It is known that subobjects are preserved by direct products, *i.e.*, if both $x: X \to X'$ and $y: Y \to Y'$ are monomorphisms, then so also is the morphism $x \times y: X \times Y \to X' \times Y'$ induced by x and y. We remark that a similar proposition holds for extremal subobjects.

LEMMA 4. *In a category which has pushouts, if both $x: X \to X'$ and $y: Y \to Y'$ are extremal monomorphisms, then so also is the morphism $x \times y: X \times Y \to X' \times Y'$ induced by x and y.*

Proof is easy and omitted here.

Now we return to the investigation of finitary objects, and show that being finitary is an intrinsic property of objects in a way.

LEMMA 5. *In a category $\underline{C}$ which has pullbacks, pushouts and co-intersections of quotient objects of any object, if an object A is not finitary, then A is not finitary under an object B which is co-generated by A.*

Proof. Assume that A is not finitary under B', and a family $\{b_\lambda: X_\lambda \to B' \mid \lambda \in \Lambda\}$ covers B' and is finitely compatible but not compatible with $\{a_\lambda: X_\lambda \to A \mid \lambda \in \Lambda\}$. Let Δ be the family of all finite subsets of Λ, then for each $M \in \Delta$ there exists an $f_M: B' \to A$ such that $a_\lambda = f_M b_\lambda$ for $\lambda \in M$. Let $e: B' \to B$ be the co-intersection of all epimorphisms $B' \to \cdot$ that factor all f_M, $M \in \Delta$.

First, we shall show that B is co-generated by A. Indeed, putting $f_M = f'_M e$, we have $\{f'_M \mid M \in \Delta\} \subset \underline{C}(B, A)$ and if $e': B \to \cdot$ is an epimorphism that factors all f'_M then $e'e$ factors all f_M and hence e. This means that e' is a left reversible epimorphism and hence an isomorphism.

Next, $\{eb_\lambda: X_\lambda \to B \mid \lambda \in \Lambda\}$ covers B. In fact, if $m: C \to B$ is an extremal monomorphism that factors all eb_λ as $eb_\lambda = mc_\lambda$, then, letting the

$$\begin{tikzcd}
X_\lambda \arrow[dr] \arrow[drr, "b_\lambda"] \arrow[ddr, "c_\lambda"'] & & \\
& D \arrow[r, "b"'] \arrow[d, "c"] & B' \arrow[d, "e"] \\
& C \arrow[r, "m"'] & B
\end{tikzcd}$$

square $DCBB'$ in the diagram above be a pullback diagram, $b: D \to B'$ is an extremal monomorphism (Lemma 1 (i)) that factors all b_λ, $\lambda \in \Lambda$. Thus b is an isomorphism, and $e = m(cb^{-1})$. Hence m is a terminal factor of an epimorphism e and an epimorphism itself. Therefore, the epimorphic extremal monomorphism m is an isomorphism.

Finally, $\{eb_\lambda \mid \lambda \in \Lambda\}$ is finitely compatible with $\{a_\lambda \mid \lambda \in \Lambda\}$, since for each $M \in \Delta$ and $\lambda \in M$, we have $a_\lambda = f_M b_\lambda = f'_M(eb_\lambda)$. However, the former is not compatible with the latter, since the existence of a $g: B \to A$ such that $a_\lambda = geb_\lambda$ for all $\lambda \in \Lambda$ immediately contradicts the assumption that $\{b_\lambda \mid \lambda \in \Lambda\}$ is not compatible with $\{a_\lambda \mid \lambda \in \Lambda\}$ q.e.d.

COROLLARY. *In a category $\underline{C}$ which is locally small and has pullbacks, pushouts, direct products and image-decompositions, if an object A is not finitary, then it is not finitary under an object which is co-generated by A.*

Proof. In the proof of the previous lemma, let $\prod A_M$ be the direct product of replicas A_M of A for $M \in \Delta$ with projections $\pi_M: \prod A_M \to A_M$, and $f: B' \to \prod A_M$ the morphism such that $f_M = \pi_M f$ for every $M \in \Delta$. Letting $f = me$ be the image-decomposition of f with $m: B \to \prod A_M$, it is similarly proved as in the previous lemma that B serves as the object under which A is not finitary, while A co-generates B by Lemma 2. q.e.d.

THEOREM 3. *Under the same assumption for* $\underline{C}$ *as in Lemma* 5, *if* A *is finitary in the full subcategory generated by all objects which are co-generated by* A, *then* A *is finitary in* $\underline{C}$.

Proof. If A is not finitary in $\underline{C}$, then by the previous lemma, A is not finitary under an object B which is co-generated by A, and there exists a family $\{b'_\lambda: X'_\lambda \to B \mid \lambda \in \Lambda\}$ which covers B and finitely compatible but not compatible with $\{a'_\lambda: X'_\lambda \to A \mid \lambda \in \Lambda\}$. Let $e_\lambda: X'_\lambda \to X_\lambda$ be the co-intersection of all epimorphisms $X'_\lambda \to \cdot$ that factor both a'_λ and b'_λ. Then we have an a_λ and a b_λ such that $a'_\lambda = a_\lambda e_\lambda$ and $b'_\lambda = b_\lambda e_\lambda$.

Now X_λ is co-generated by A. Indeed, assume that $x: X_\lambda \to C$ is an epimorphism that factors all morphism in $\underline{C}(X_\lambda, A)$. Let the square $X_\lambda CDB$ is in the diagram below be a pushout diagram, then $b: B \to D$

$$\begin{tikzcd}
X'_\lambda \arrow[dr, "e_\lambda"] & & & \\
& X_\lambda \arrow[r, "b_\lambda"] \arrow[d, "x"'] & B \arrow[d, "b"] \arrow[ddr, "f"] & \\
& C \arrow[r, "c"] \arrow[drr, "g"'] & D \arrow[dr, "d"] & \\
& & & A
\end{tikzcd}$$

is an epimorphism. Since for any $f: B \to A$, $fb_\lambda: X_\lambda \to A$ is factored by x as $fb_\lambda = gx$, there exists a $d: D \to A$ such that $f = db$ and $g = dc$. Thus the epimorphism b factors all f in $\underline{C}(B, A)$, and since B is co-generated by A, it is an isomorphism. Hence $b_\lambda = (b^{-1} c)x$ and x factors both b_λ and a_λ which is a priori in $\underline{C}(X_\lambda, A)$. This implies that xe_λ factors both a'_λ and b'_λ and hence e_λ. Thus x is a left reversible epimorphism, and hence an isomorphism.

It is obvious that $\{b_\lambda \mid \lambda \in \Lambda\}$ covers B, since an extremal monomorphism $B' \to B$ that factors all b_λ also factors all $b'_\lambda (= b_\lambda e_\lambda)$.

Since $e_\lambda: X' \to X$ is an epimorphism, for any $f: B \to A$, the equality $a_\lambda = fb_\lambda$ is equivalent to $a_\lambda e_\lambda = fb_\lambda e_\lambda$ i.e., $a'_\lambda = fb'_\lambda$. From this it immediately follows that $\{b_\lambda \mid \lambda \in \Lambda\}$ is finitely compatible, but not compatible with $\{a_\lambda \mid \lambda \in \Lambda\}$. q.e.d.

COROLLARY 1. *Under the same condition for* $\underline{C}$ *as in the corollary of Lemma* 5, *if an object* A *is finitary in the full subcategory generated by all objects which are co-generated by* A, *then* A *is finitary in* $\underline{C}$.

Proof. In the previous proof, let p_λ: $X'_\lambda \to B \times A$ be the morphism such that $b'_\lambda = \pi_B p_\lambda$ and $a'_\lambda = \pi_A p_\lambda$, where π_B: $B \times A \to B$ and π_A: $B \times A \to A$ are the projections to the components, and $p_\lambda = m_\lambda e_\lambda$ the image-decomposition of p_λ with m_λ: $X_\lambda \to B \times A$. Then it is similarly proved as in the proof of Theorem 3 that $\{\pi_B m_\lambda \mid \lambda \in \Lambda\}$ covers B and is finitely compatible but not compatible with $\{\pi_A m_\lambda \mid \lambda \in \Lambda\}$, while, referring to Lemma 4 and Lemma 1 (iii), it is seen that X_λ is an extremal subobject of a direct power of A. q.e.d.

Particularly, if we understand the left completeness of a category and the left closedness of a subcategory of a category in the sense of Isbell [2], it follows from 3.3.*b* of [2] that:

COROLLARY 2 *In a left complete locally small category* $\underline{C}$ *which has pushouts, if an object* A *is finitary in the full left closure of* $\{A\}$, *then it is finitary in* $\underline{C}$.

§ 2. **Ultraproducts.** Let Ξ be a set and A_ξ an object assigned to each $\xi \in \Xi$. From here on, the projection from the direct product $\prod_{\xi \in \Xi} A_\xi$ to its component A_ξ will be denoted by π^Ξ_ξ. When $\Xi' \subset \Xi$, there is a morphism $\prod_{\xi \in \Xi} A_\xi \to \prod_{\xi \in \Xi'} A_\xi$, which is also denoted by $\pi^\Xi_{\Xi'}$, such that $\pi^\Xi_\xi = \pi^{\Xi'}_\xi \pi^\Xi_{\Xi'}$ for every $\xi \in \Xi'$. $\pi^\Xi_{\Xi'}$ is also called a projection.

The definition of ultraproducts in terms of categories was given in [4] as follows:

Definition 2. Let Γ be a set and Φ a filter over Γ, i.e., a family of non-void subsets of Γ such that (i) $\Xi, \Xi' \in \Phi$ implies $\Xi \cap \Xi' \in \Phi$ and (ii) $\Xi \in \Phi$ and $\Xi \subset \Xi'$ imply $\Xi' \in \Phi$. Assume that an object A_ξ is assigned to each $\xi \in \Gamma$. The diagram in $\underline{C}$ which consists of all $\prod_{\xi \in \Xi} A_\xi$ with $\Xi \in \Phi$ as objects and all $\pi^\Xi_{\Xi'}$: $\prod_{\xi \in \Xi} A_\xi \to \prod_{\xi \in \Xi'} A_\xi$ with $\Xi' \subset \Xi$ and $\Xi, \Xi' \in \Phi$ as morphisms is called a *product system relative* to the filter Φ. The right limit, if exists, of the product system is called the *reduced product* of the family $\{A_\xi \mid \xi \in \Gamma\}$ relative to Φ, and denoted by $\prod_{\xi \in \Gamma} A_\xi / \Phi$. When $A_\xi = A$ for all $\xi \in \Gamma$, it is called the *reduced power* of A relative to Φ, and denoted by A^Γ/Φ. When Φ is maximal, it is called an *ultrafilter* of Γ, and $\prod_{\xi \in \Gamma} A_\xi/\Phi$ and A^Γ/Φ are respectively called an *ultraproduct* and an *ultrapower* relative to Φ. The canonical injection $\prod_{\xi \in \Xi} A_\xi \to \prod_{\xi \in \Gamma} A_\xi/\Phi$ is also denoted by π^Ξ. For a direct power A^Ξ, there is the so-called diagonal morphism $A \to A^\Xi$, which is denoted by d_Ξ, such that $\pi^\Xi_\xi d_\Xi$: $A \to A_\xi$ is the identity for every $\xi \in \Xi$. Obviously $d_{\Xi'} = \pi^\Xi_{\Xi'} d_\Xi$ for $\Xi' \subset \Xi$, and $\Xi, \Xi' \in \Phi$, and hence there is a d: $A \to A^\Gamma/\Phi$ such that $d=\pi^{\Xi}d_{\Xi}$ for all $\Xi\in\Phi$. d is also called the *diagonal morphism* to the reduced power.

Those are a natural extension of usual ultraproducts in model theory and the concomitant notions (see [3], for example). However, the general application of the definition above brings about pathological phenomena and many of useful properties of ultraproducts are no more retained. For instance, it was pointed out in [4] that, in the category of hausdorff spaces with morphisms of continuous mappings, an ultraproduct is always reduced to a singleton space unless the ultrafilter is principal. Thus the diagonal morphism d of an ultrapower is no more an embedding, while it is one of the fundamental characters in model theory that d is always elementary. Kochen [3] called d the canonical imbedding. The reason we avoided the term is that it is not necessarily an embedding as stated above.

However, it is natural to find an interest in the problem when d may be an embedding, or an extremal monomorphism in the diction of categories, and here one finds some connection with the notion of finitary objects.

THEOREM 4. *In a category $\underline{C}$ which is locally small and small complete to the both sides, if the diagonal morphism $d\colon A\to A^{\Gamma}/\Phi$ is an extremal monomorphism for any set Γ and any filter Φ over it, then A is finitary.*

Proof. Assume that the family $\{b_\lambda\colon X_\lambda\to B \mid \lambda\in\Lambda\}$ covers B and is finitely compatible with $\{a_\lambda\colon X_\lambda\to A \mid \lambda\in\Lambda\}$. Put $\Gamma=\underline{C}(B, A)$, which is assumed small. Let Δ be the set of all finite subsets of Γ, and, for each $M\in\Delta$, $\tilde{M}$ be the set of all $f\in\Gamma$ such that $a_\lambda=fb_\lambda$ for all $\lambda\in M$. $\tilde{M}$ is not void for any $M\in\Delta$, and the collection $\{\tilde{M}\mid M\in\Delta\}$ makes a filter over Γ. Let it be extended to a maximal filter Φ. With each $f\in\Gamma$ we associate a replica A_f of A and construct the ultrapower A^{Γ}/Φ. For $M\in\Delta$ an $f_M\colon B\to\prod_{f\in\tilde{M}}A_f$ is determined so that $\pi_f^{\tilde{M}}f_M=f$ for every

$$\begin{tikzcd}
X \arrow[r, "b_\lambda"] \arrow[dd, "a_\lambda"'] & B \arrow[d, "f_M"] \arrow[dd, bend right=40, "f"'] \arrow[dr, "t"] & \\
& \prod_{f\in\tilde{M}}A_f \arrow[d, "\pi_f^{\tilde{M}}"] \arrow[r, "\pi^{\tilde{M}}"'] & A^{\Gamma}/\Phi \\
A \arrow[r, equal] & A_f \arrow[ur, "d"'] &
\end{tikzcd}$$

$f \in \tilde{M}$. Put $t=\pi^{\tilde{M}} f_M\colon B \to A^{\Gamma}/\Phi$. Then since $f_{M'}=\pi^{\tilde{M}}_{\tilde{M}'} f_M$ for $M' \subset M$, t is determined independently of the choice of $M \in \Delta$.

First we establish the commutativity $tb_\lambda = da_\lambda$ for every $\lambda \in \Gamma$. Indeed, let M be any set with $\lambda \in M \in \Delta$. Then by the definition of $\tilde{M}$, for any $f \in \tilde{M}$ we have $a_\lambda = fb_\lambda = \pi^{\tilde{M}}_f f_M b_\lambda$. This means $d_{\tilde{M}} a_\lambda = f_M b_\lambda$ and hence $da_\lambda = \pi^{\tilde{M}} d_{\tilde{M}} a_\lambda = \pi^{\tilde{M}} f_M b_\lambda = tb_\lambda$.

$$\begin{tikzcd}
X_\lambda \arrow[dr] \arrow[drr, "b_\lambda"] \arrow[ddr, "a_\lambda"'] & & \\
& C \arrow[r, "b"'] \arrow[d, "a"] & B \arrow[d, "t"] \\
& A \arrow[r, "d"'] & A^{\Gamma}/\Phi
\end{tikzcd}$$

Let the square $CA(A^{\Gamma}/\Phi)B$ in the diagram above be a pullback diagram, then, since d is an extremal monomorphism by assumption, so also is $b\colon C \to B$ (Lemma 1 (i)), and the commutativity $da_\lambda = tb_\lambda$ just established implies that b factors all b_λ. Hence b is an isomorphism and we have $a_\lambda = (ab^{-1})b_\lambda$ for every $\lambda \in \Lambda$ q.e.d.

Remark. Observe that the only property of extremal monomorphisms crucial in the proof of Theorem 4 is the one mentioned as Lemma 1 (i). This property may be referred to as "*extremal monomorphisms are preserved by pullbacks* (in a category which has pushouts)". Therefore, if we take a class Ω of monomorphisms which are preserved by pullbacks (in a category with proper conditions), then, Ω-monomorphisms, morphisms in Ω, can take the rôle of extremal monomorphisms in all statements, obtaining a theorem parallel to Theorem 4. For example, assume Ω consists of all monomorphisms, then a lemma similar to Lemma 1 (i) holds without the condition that $\underline{C}$ has pushouts. Now a family $\{b_\lambda\colon X_\lambda \to B \mid \lambda \in \Lambda\}$ may be said to cover B in wider sense, if there is no proper monomorphism $\cdot \to B$ that factors all b_λ. Replacing the notion of covering in Definition 1 with that of covering in wider sense, we can define "A is *wide-finitary*" in place of "A is finitary". Thus we have the theorem: *In a locally small both-side small complete category $\underline{C}$, if $d\colon A \to A^{\Gamma}/\Phi$ is monomorphic for any set Γ and any filter Φ over it, then A is wide-finitary.* Further, it is observed that most of the theorems in § 1 also hold for wide-finitary objects under proper substitution of terms and slight modifications of conditions. And this situation is all the same when we take the class of all equalizers (which are also preserved by pullbacks without the pushout condition) for Ω. There may be particular interests in each of those various finitarities and important differences between them. However, the finitarity given in Definition 1 seems most essential in the application to concrete (in the literal sense) categories, and we miss good examples of other notions of modified finitarities.

By the way, the converse of Theorem 4 does not seem true and the characterization for A so that d be an extremal monomorphism for any Γ and Φ is still open.

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